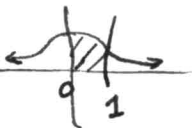


Name: Solutions

Section: _____

Net Area and Integrals



1. The integral $\int_a^b e^{-x^2} dx$ is important in statistics,¹ but it is infamously hard to compute. Many statistics textbooks include a table which lists the value of the integral for different values of a and b . We will use Riemann Sums to generate one of these approximations.

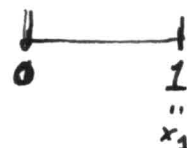
(a) Express the integral $\int_0^1 e^{-x^2} dx$ as the limit of its Right Riemann Sums.

$$\int_0^1 e^{-x^2} dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n e^{-(x_i)^2} \cdot \Delta x \quad \text{where } \Delta x = \frac{1-0}{n} \text{ and } x_i = a + i \cdot \Delta x$$

(b) Approximate $\int_0^1 e^{-x^2} dx$ using Right Sums and $n = 1$. Use a calculator to simplify.

$$R_1 = f(x_1) \cdot \Delta x \quad \Delta x = \frac{1-0}{1} \quad x_1 = 1$$

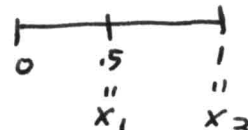
$$= e^{-(1)^2} \cdot 1 \approx .36788$$



(c) Approximate $\int_0^1 e^{-x^2} dx$ using Right Sums and $n = 2$. Use a calculator to simplify.

$$R_2 = f(x_1) \Delta x + f(x_2) \Delta x \quad \Delta x = \frac{1-0}{2} = .5$$

$$= e^{-(.5)^2} (.5) + e^{-(1)^2} (.5) \approx .59334$$

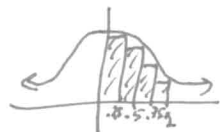
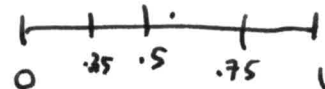


(d) Approximate $\int_0^1 e^{-x^2} dx$ using Right Sums and $n = 4$. Use a calculator to simplify.

$$R_4 = f(x_1) \Delta x + f(x_2) \Delta x + f(x_3) \Delta x + f(x_4) \Delta x \quad \Delta x = \frac{1-0}{4} = .25$$

$$= e^{-(.25)^2} (.25) + e^{-(.5)^2} (.25) + e^{-(.75)^2} (.25) + e^{-(1)^2} (.25)$$

$$\approx .66397$$



$$\Delta x = \frac{1-0}{8} = .125$$

(e) Approximate $\int_0^1 e^{-x^2} dx$ using Right Sums and $n = 8$. Use a calculator to simplify.

$$R_8 = f(x_1) \Delta x + f(x_2) \Delta x + f(x_3) \Delta x + f(x_4) \Delta x + f(x_5) \Delta x + f(x_6) \Delta x + f(x_7) \Delta x + f(x_8) \Delta x$$

$$= (.125) \left[e^{-(.125)^2} + e^{-(.25)^2} + e^{-(.375)^2} + e^{-(.5)^2} + e^{-(.625)^2} + e^{-(.75)^2} + e^{-(.875)^2} + e^{-(1)^2} \right]$$

$$\approx .70636$$

(f) How do these compare to the correct value of $\int_0^1 e^{-x^2} dx = .7468241 \dots$?

they get closer as n gets bigger.

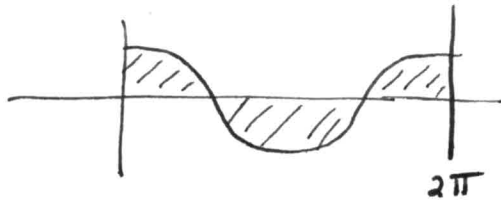
¹This and other similar integrals are needed to compute the probability of events that follow a normal distribution. See, for example, http://en.wikipedia.org/wiki/Standard_normal_distribution#Cumulative_distribution.

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2. What is the graphical meaning of $\int_0^{2\pi} \cos(x) dx$? Compute this area geometrically.

the net area



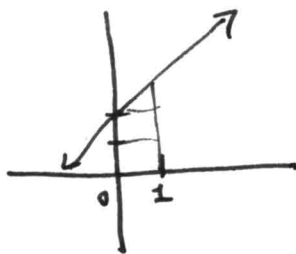
These areas cancel

so

~~the~~ integral = net area = 0

3. What is the graphical meaning of $\int_0^1 x + 2 dx$? Compute this area geometrically.

the net area below

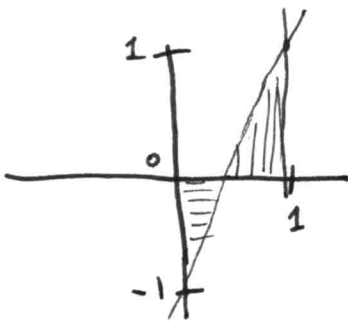


$$\text{net area} = 1 + 1 + \frac{1}{2} = \frac{5}{2}$$

4. What is the graphical meaning of $\int_0^1 2x - 1 dx$? Compute this area geometrically.

the net area

$$= \left(-\frac{1}{4}\right) + \left(\frac{1}{4}\right) = 0$$

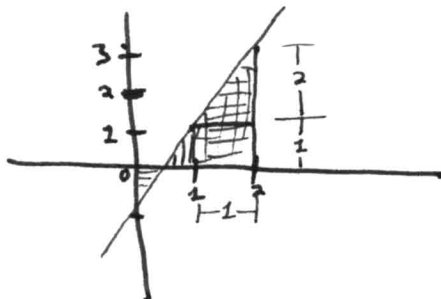


5. What is the graphical meaning of $\int_0^2 2x - 1 dx$? Compute this area geometrically.

the net area

$$\text{net area} = \left(-\frac{1}{4}\right) + \left(\frac{1}{4}\right) + 1 + \frac{1}{2}(2)(1)$$

$$= 1 + 1 = 2$$



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6. Determine if the following are **True or False**

(a) $\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$ True False

(b) $\int_a^b (c \cdot f(x)) dx = c \cdot \left(\int_a^b f(x) dx \right)$ True False

(c) $\int_a^b (f(x) - g(x)) dx = \int_a^b f(x) dx - \int_a^b g(x) dx$ True False

7. Suppose that $\int_0^3 f(x) dx = 5$, $\int_4^3 f(x) dx = 2$, and $\int_4^6 f(x) dx = 10$. Find

(a) $\int_0^4 f(x) dx = \int_0^3 f(x) dx + \int_3^4 f(x) dx = 5 + (-2) = 3$

(b) $\int_3^6 f(x) dx = \int_3^4 f(x) dx + \int_4^6 f(x) dx = (-2) + 10 = 8$

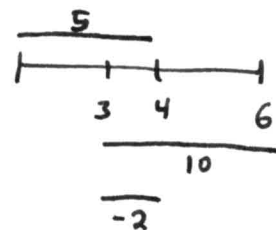
(c) $\int_0^6 f(x) dx = \int_0^3 f(x) dx + \int_3^6 f(x) dx = 5 + 8 = 13$

8. Suppose that $\int_0^4 f(x) dx = 5$, $\int_3^4 f(x) dx = -2$, and $\int_3^6 f(x) dx = 10$. Find

(a) $\int_0^3 f(x) dx = \int_0^4 f(x) dx - \int_3^4 f(x) dx = 5 - (-2) = 7$

(b) $\int_4^6 f(x) dx = \int_3^6 f(x) dx - \int_3^4 f(x) dx = 10 - (-2) = 12$

(c) $\int_0^6 f(x) dx = \int_0^4 f(x) dx + \int_4^6 f(x) dx = 5 + 12 = 17$



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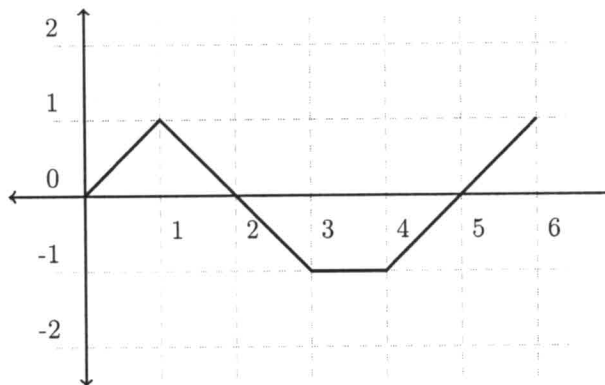
The Fundamental Theorems

1. Compute $\frac{d}{dx} \left[\int_5^x t^2 + 1 dt \right] = x^2 + 1$

2. Compute $\frac{d}{dx} \left[\int_1^x \sin(5t) dt \right] = \sin(5x)$

3. Compute $\frac{d}{dx} \left[\int_{-3}^x \sin(\cos(e^t)) dt \right] = \sin(\cos(e^x))$

4. Suppose that the function $f(x)$ is given by the following graph.



Let $A(x) = \int_0^x f(t) dt$. Compute the following

(a) $A(1) = \frac{1}{2}$

(b) $A(2) = 1$

(c) $A(4) = -\frac{1}{2}$

(d) $A'(1) = 1$

(e) $A'(2) = 0$

(f) $A'(4) = -1$

$$A'(x) = \frac{d}{dx} \left[\int_0^x f(t) dt \right] = f(x)$$

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Computing Integrals Quickly

$$\begin{aligned}
 1. \int [\sin(x) + e^x - 2 \cos(x) + 6x^2] dx \\
 &= \int \sin(x) dx + \int e^x dx - 2 \int \cos(x) dx + 6 \int x^2 dx \\
 &= \int -\cos(x) + e^x - 2 \cdot \sin(x) + 6 \cdot \frac{x^3}{3} + C \\
 &= -\cos(x) + e^x - 2\sin(x) + 2x^3 + C
 \end{aligned}$$

$$2. \int_0^1 \frac{1}{x^2+1} dx = \tan^{-1}(x) \Big|_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$$

$$\tan^{-1}(1) = x \Leftrightarrow 1 = \tan(x) \Leftrightarrow x = \frac{\pi}{4}$$

$$\tan^{-1}(0) = x \Leftrightarrow 0 = \tan(x) \Leftrightarrow x = 0$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$\begin{aligned}
 3. \int \frac{x^2+x+1}{x} dx &= \int \frac{x^2}{x} dx + \int \frac{x}{x} dx + \int \frac{1}{x} dx \\
 &= \int x dx + \int 1 dx + \int \frac{1}{x} dx \\
 &= \frac{x^2}{2} + x + \ln|x| + C
 \end{aligned}$$

$$\begin{aligned}
 4. \int_{-1}^1 (x^2+3)(x-1) dx &= \int_{-1}^1 x^3 + 3x - x^2 - 3 dx \\
 &= \left[\frac{x^4}{4} + \frac{3x^2}{2} - \frac{x^3}{3} - 3x \right]_{-1}^1 = \left(\frac{1}{4} + \frac{3}{2} - \frac{1}{3} - 3 \right) - \left(\frac{1}{4} + \frac{3}{2} - \frac{1}{3} - 3 \right) \\
 &= \frac{1}{4} - \frac{1}{4} + \frac{3}{2} - \frac{3}{2} - \frac{1}{3} - \frac{1}{3} - 3 - 3 \\
 &= -\frac{2}{3} - 6
 \end{aligned}$$

$$5. \int \frac{x^{3/2} + \sqrt{x} + 1}{\sqrt{x}} dx$$

$$= \int \frac{x^{\frac{3}{2}}}{\sqrt{x}} dx + \int \frac{\sqrt{x}}{\sqrt{x}} dx + \int \frac{1}{\sqrt{x}} dx$$

$$= \int \frac{x^{\frac{3}{2}}}{x^{\frac{1}{2}}} dx + \int 1 dx + \int x^{-\frac{1}{2}} dx$$

$$= \int x^{\left(\frac{3}{2}-\frac{1}{2}\right)} dx + \int 1 dx + \int x^{-\frac{1}{2}} dx$$

$$= \int x dx + x + \frac{x^{\frac{1}{2}}}{\frac{1}{2}}$$

$$= \frac{x^2}{2} + x + 2\sqrt{x} + C$$

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6. $\int x \sqrt{x^2+3} dx$ \xrightarrow{u} $\int \sqrt{u} \frac{du}{2} = \frac{1}{2} \int u^{\frac{1}{2}} du$
 $= \frac{1}{2} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{u^{\frac{3}{2}}}{3} + C$
 $= \frac{(x^2+3)^{\frac{3}{2}}}{3} + C$

$u = x^2+3$
 $\frac{du}{dx} = 2x$
 $du = 2x dx$
 $\frac{du}{2} = x dx$

7. $\int \frac{1}{x+1} dx$ \xrightarrow{u} $\int \frac{1}{u} du = \ln|u| + C$
 $= \ln|x+1| + C$

$u = x+1$
 $\frac{du}{dx} = 1$
 $du = dx$

8. $\int \sin(5x) + 1 dx = \int \sin(5x) dx + \int 1 dx$
 $= \frac{\cos(5x)}{5} + x + C$
 $= \frac{\cos(5x)}{5} + x + C$

$u = 5x$
 $\frac{du}{dx} = 5$
 $\frac{du}{5} = dx$

9. $\int \cos(x) \cdot e^{\sin(x)} dx = \int e^u du = e^u + C$
 $= e^{\cos(x)} + C$

$u = \sin(x)$
 $\frac{du}{dx} = \cos(x)$
 $du = \cos(x) dx$

10. $\int \left[\frac{1}{x^2+1} + \frac{1}{x+1} \right] dx = \int \frac{1}{x^2+1} dx + \int \frac{1}{x+1} dx$

$= \tan^{-1}(x) + \ln|x+1| + C$

$= \tan^{-1}(x) + \ln|x+1| + C$

$u = x+1$
 $\frac{du}{dx} = 1$
 $du = dx$
 $\int \frac{1}{u} du = \ln|u| + C$